B.Sc Part II Physics (Hons)
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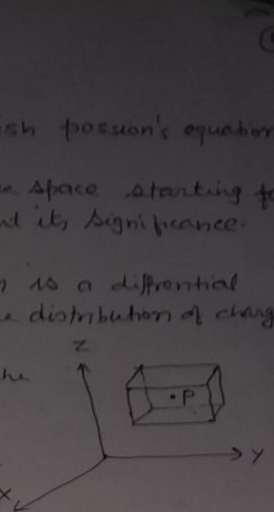
Q (a) What is Poisson's equation? Establish possion's equation for a space distrubution of charge.

(b) Establish Laplace's equation for free space starting to from Gauss's law and discuss about its significance.

Ans (a) **Poisson's equation:→**

Poisson's equation is a diffrential equation which is obtained for a space distribution of charge on the basis of Gauss's theorem.



Let OX, OY and OZ are the Cartesean axes and P is a point whose co-ordinates are x, y, z. A parallel and eque-angle octogonal is drawn around the point P. The parallel arms of x, y & z axes are Δx ~~by~~, Δy & Δz respectively. σ is the surface charged density. Again suppose that electric displacement or induction of p point is D and K is permitivity of medium. The Component of electric Induction D is OX, OY and OZ direction is D_x, D_y & D_z. Electric intensity is E and its components are E_x, E_y & E_z respectively.

Due to x-component the total normal electric intensity D is on the infront face which is parallel to the surface YZ, whose area $= \Delta y \, \Delta z$

In the posetion of centre of orthogonal parallel to point P, the co-ordinate of centre $= \left[\left(x + \frac{\Delta x}{2}\right), y, z\right]$

Therefore rate of charged with respect to x of $D_x = \frac{\partial D_x}{dx}$

∴ The amount of x component of electric induction D is on centre $= \left[D + \frac{\partial D_x}{\delta x} \cdot \frac{\Delta x}{2}\right]$

So the total normal electric induction on the face

$$= \left[D_x + \frac{\partial D_x}{\partial x} \cdot \frac{\Delta x}{2}\right] \Delta y, \Delta z$$

Therefore, the total electric induction in the parallel direction of OX

$$= \left[D_x + \frac{\partial D_x}{\partial x} \cdot \frac{\Delta x}{2}\right] \Delta y . \Delta z - \left[\Delta x - \frac{\partial D_x . \Delta x}{\partial x . 2}\right] \Delta y \, \Delta z$$

$$= \frac{\delta D_x}{\partial x} \Delta x . \Delta y . \Delta z$$

Similarly, the total normal electric induction on the ⊥ faces of OY + OZ axis are

$$\frac{\partial D_y}{\partial y}\Delta x \Delta y \Delta z \text{ and } \frac{\partial D_z}{\partial z}\Delta x \Delta y \Delta z$$

Therefore, total normal electric induction on the surface of parallel ~~el~~ octogonal

$$= \left(\frac{\delta D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\delta D_z}{\partial z}\right)\Delta x \Delta y \Delta z$$

According to Gauss's theorem, we know that total normal electric induction is equal to 4π temes of its total charge.

$$\therefore \left(\frac{\delta D_x}{\partial x} + \frac{\delta D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta x \Delta y \Delta z = 4\pi\sigma \,\Delta x \Delta y \Delta z$$

Here σ = Surface charged density

$$\text{or } \left(\frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = 4\pi\sigma \quad \text{------ (I)}$$

$$\because D = KE \quad \text{------ (II)}$$

$$\therefore \left(\frac{\delta E_x}{\partial x} + \frac{\delta E_y}{\partial y} + \frac{\delta E_z}{\delta z}\right) = \frac{4\pi\sigma}{K}$$

when eq^n (I) and (II) is in vector form, we get

$$\text{------ (III)}$$

$$\operatorname{div}\vec{D} = \vec{\nabla}\cdot\vec{D} = 4\pi\sigma \quad \text{------ (IV)}$$

$$\operatorname{div}\vec{E} = \vec{\nabla}\cdot\vec{E} = \frac{4\pi\sigma}{K} + \frac{\sigma}{\varepsilon_0} \quad \text{------}$$

Suppose that potential on point P in parallel octogonal $= V(x, y, z)$, then

$$E_x = -\frac{\delta V}{\partial x} \cdot E_y = \frac{\delta V}{\partial y} \text{ and } E_z = \frac{-\delta V}{\delta z}$$

Therefore the eq^n (II) may be written in following form

$$\frac{\partial^2 V}{\delta x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\delta z^2} = \frac{-4\pi\sigma}{K} = \frac{-\sigma}{\varepsilon_0} \quad \text{------ (V)}$$

which is in the form of vector,

$$\nabla^2 V = -\frac{4\pi\sigma}{K} = -\frac{\sigma}{\varepsilon_0} \quad \text{------ (VI)}$$

$$\text{or } \operatorname{div}\operatorname{grad} V = -\frac{\sigma}{\varepsilon_0}$$

The eq^n (V) and (VI) are called Poisson's equation

Again let $\sigma = 0$

Then $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$ .... (VII)

or $\nabla^2 y = 0$ .... (VIII)

or div grad $V = 0$

Eq^n (VII) and (VIII) are called Laplace's eq^n for free space.

(b) It has the following aspects:→

1. If $V(x, y, z)$ is a solution of Laplace's eq^n, then the average value of V over the surface of any system is sphere is equal to the value of V at centre of sphere.
2. If $V_1, V_2 \ldots V_K$ are solution of Laplace's equation then $V = A_1V_1 + A_2V_2 + \ldots A_KV_K$ is also a solution where A's are orbitrary constants
3. If $\vec{V}$ is a solution of Laplace's equation, then all partial derivatives of V w.r.to one or more of the co-ordenates x, y, z such as $\frac{\delta v}{\delta x}, \frac{\partial^2 v}{\delta x^2}, \frac{\partial^2 v}{\delta x \delta y}$ etc are sol^n
4. A defenite solution of Laplace's eq^n can be obtained only where some boundary conditions are given.
5. With given boundary conditions Laplace's equation has one and only one solution.